The assumption of a sharp temperature change is only justifiable, however, when the gage is not very small.

This analysis confirms the importance of measuring only the initial response of a calorimeter gage whose thermal properties differ considerably from those of the mounting surface.

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Limits on the Damping of Two-Body Gravitationally Oriented Satellites

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This paper considers the conjecture that the settling time of a strictly gravity-gradient attitude control system is limited to be of the order of the settling time of a critically damped dumbbell. The conjecture is shown to be true for a certain class of gravity-gradient systems and, in particular, for the gravity-gradient systems thus far proposed. It also is shown that gravity-gradient systems outside this class may have arbitrarily fast settling times. However, it is suggested that reliable mechanization of such rapidly settling systems may be difficult.

I. Introduction

In a previous paper,[1] the author considered the small motion damping of a two-body gravitationally oriented satellite of the type proposed by Kamm.[2] It was pointed out that this was one of several passive or semipassive gravitational schemes being considered for very reliable, long-life satellites. The common feature of all the schemes is an auxiliary inertia, either a gyro or a second rigid body, which is attached to the satellite through a dissipative joint.

For the satellite considered in Ref. 1, a bound was found on the pitch "asymptotic settling time," that is, the $1/e$ settling time of the most lightly damped mode. This was found to be $5^{1/4}/2\pi(3)^{1/2} = 0.137$ orbits. In this satellite, only a simple spring-dashpot combination was assumed between the two inertias. One immediately thinks of improving the settling time by the use of more sophisticated control, say by employing feedback. On the other hand, the bound on the $1/e$ pitch asymptotic settling time of the roll-vee gyro system of Refs. 3 and 4 is $1/(2\pi) = 0.159$ orbits—of the same order as that of the system in Ref. 1. One might conjecture that a natural bound of this order of magnitude generally exists for purely gravity-gradient schemes.

For example, suppose that the auxiliary inertia is a second rigid body. The satellite is desired stable with respect to a rotating, earth-pointing reference frame. In this frame, the satellite's natural frequency can be no higher than $(3)^{1/2}\Omega$ (Ω is the orbital frequency), corresponding to the natural frequency of a dumbbell-shaped body. Likewise, the natural frequency of any auxiliary fluid or rigid body inertia system also is less than $(3)^{1/2}\,\Omega$, and so the basic system one starts with has the lumped representation shown in Fig. 1, with two inertia systems of limited frequencies. (For convenience, a lineal rather than a rotatory model is shown.) Only if the satellite grabs onto the rotating, earth-pointing reference frame by some means other than gravity gradient can these frequencies be raised. One now provides torques between the inertias to damp the system as rapidly as possible. However, these torques are applied only *between* the inertias. It would seem likely, therefore, that the limited natural frequencies of the satellite and the auxiliary inertia would set the time scale of the oscillation. One thus might conjecture that it would be difficult to attain settling times much faster than $1/[2\pi(3)^{1/2}] = 0.092$ orbits, corresponding to the most rapid (critical) damping of a single-degree-of-freedom system with the limiting natural frequency of $(3)^{1/2}\Omega$.

It is shown in this paper that the conjecture is true for a certain class of linear systems. Systems in this class have resistive velocity-dependent torques (as defined in the next section). In addition, torques proportional to the bodies' displacements are applied between the inertias. For example, all systems with a single viscous damper and displacement proportional torques between the inertias fall within this class. The class includes, in particular, all the gravity-gradient schemes considered in Refs. 1, 2, and 5. The compliant dumbbell analyzed by Paul[6] is not of this class. However, by the methods presented, the conjecture easily is shown to hold for Paul's system as well. The conjecture thus, in fact, is true for all the gravity-gradient schemes proposed in Refs. 1–6.

Hence, to obtain a two-body gravity-gradient satellite that damps down substantially faster than these proposed systems, one must search outside the class for which the conjecture holds. Indeed, as shown in this paper by an example, it is possible to attain arbitrarily fast settling times outside of this class. However, the system shown in this paper to have arbitrarily rapid settling times also is shown to be intolerably sensitive to changes in system parameters. It goes without saying that the mechanization of any practical damping system must be reliably long-life so as not to negate the basic gravity-gradient reliability. Whether this is possible for a system outside the class considered is an open question.

II. Systems for Which the Conjecture Holds

Fourth-order system

Consider the system shown in Fig. 1. This is a schematic of the pitch motion of a two-body, gravity-gradient system. The bodies A_1 and A_2 are assumed linked at their mass centers† so that the gravity-gradient spring, $k_1 = 3(B_1 - C_1)\Omega^2$, acts between the satellite of inertia A_1 and ground of the rotating reference frame. (B_1 and C_1 are principal inertias of the satellite.) Likewise, the gravity-gradient spring $k_2 = 3(B_2 - C_2)\Omega^2$ acts between the auxiliary inertia and ground. Let a torque T that is a linear function of the velocities and displacements act between the inertias A_1 and A_2:

$$T = a_1q_1 - a_2q_2 + c_1\dot{q}_1 - c_2\dot{q}_2$$

Depending on the values of a_1, a_2, c_1, and c_2, this form of the torque can represent a variety of mechanizations. For

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† As indicated in a footnote of Ref. 1, if the mass centers are noncoincident as in Kamm's vertistat, only a trivial change in the differential equations occurs. In the present case, this does not affect the results obtained.

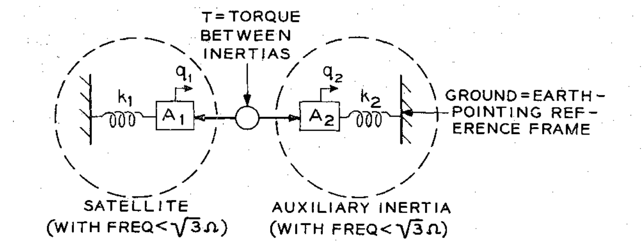

Fig. 1 Lineal lumped model of a two-body, gravity-gradient satellite (all inertias, springs, etc. are rotatory in the actual system)

example, some possibilities are:

1) $c_1 = c_2 = c$, viscous damper between A_1 and A_2.
2) $a_1 = a_2 = k$, spring of spring rate k between A_1 and A_2.
3) $a_1 = k$, $a_2 = 0$, the satellite displacement is sensed and fed back between A_1 and A_2 as a torque.
4) $a_1 = k_f + k_g$, $a_2 = k_g$, combination of 2 and 3.

The differential equations of motion for the system of Fig. 1 are

$$\begin{aligned} A_1\ddot{q}_1 + 3(B_1 - C_1)\Omega^2 q_1 &= -(c_1\dot{q}_1 - c_2\dot{q}_2) - (a_1q_1 - a_2q_2) \\ A_2\ddot{q}_2 + 3(B_2 - C_2)\Omega^2 q_2 &= (c_1\dot{q}_1 - c_2\dot{q}_2) + (a_1q_1 - a_2q_2) \end{aligned} \qquad (1)$$

Substituting $q_1 = Q_1e^{pt}$, $q_2 = Qe^{pt}$ into these equations and setting the resulting determinant to zero, one obtains the characteristic equation of the system:

$$A_1A_2p^4 + (c_1A_2 + c_2A_1)p^3 + [(k_1 + a_1)A_2 + (k_2 + a_2) \times A_1]p^2 + (c_1k_2 + c_2k_1)p + k_1k_2 + a_2k_1 + a_1k_2 = 0 \qquad (2)$$

Here k_1 and k_2 are the gravity-gradient springs:

$$k_1 = 3(B_1 - C_1)\Omega^2 \qquad k_2 = 3(B_2 - C_2)\Omega^2$$

Consider first the case when all of the coefficients of p^n in Eq. (2) are nonzero and positive. Let the farthest right roots of Eq. (2) occur on the line $Re\ p = -D$. In Ref. 7 it is shown that, for a real polynomial with positive coefficients and negative real roots

$$p^n + \alpha_1p^{n-1} + \ldots + \alpha_n = 0$$

the following bound holds:

$$D^{m-k} \leq \frac{\alpha_m/\binom{n}{m}}{\alpha_k/\binom{n}{k}} \qquad (m > k) \qquad (3)$$

Applying this inequality to the coefficients of p^3 and p, one gets

$$D^2 \leq \frac{3\Omega^2[c_1(B_2 - C_2) + c_2(B_1 - C_1)]}{c_1A_2 + c_2A_1} \qquad (4)$$

On the other hand, the moments of inertia of a rigid body have the well-known property that the sum of two principal inertias is greater than the third:

$$A_1 + C_1 \geq B_1 \qquad A_2 + C_2 \geq B_2$$

or

$$B_1 - C_1 \leq A_1 \qquad B_2 - C_2 \leq A_2 \qquad (5)$$

Consider the case $c_1 \geq 0$, $c_2 \geq 0$. [Since the coefficients of (2) are assumed nonzero, $c_1 = c_2 = 0$ is excluded.] This corresponds to velocity-dependent torques applied between the bodies which are resistive in the sense that the torque $-c_1\dot{q}_1$ then opposes the motion of body 1 and the torque $-c_2\dot{q}_2$ opposes the motion of body 2. In this case it follows immediately from (5) and (4) that

$$D \leq (3)^{1/2}\Omega \qquad (6)$$

Hence, the asymptotic decay rate D of systems within this class is bounded by the natural frequency of a dumbbell-shaped body. Likewise, the asymptotic settling time, $t_s = \Omega/2\pi D$, is bounded by $t_s = 1[2\pi(3)^{1/2}]$ orbits, and the conjecture is true.

It easily is verified that bound (6) is achieved if $c_1/A_1 = 0$, $c_2/A_2 = 4(3)^{1/2}\Omega$, $a_1/A_1 = 6\Omega^2$, $a_2/A_2 = 12\Omega^2$, $k_1/A_1 = 3\Omega^2$, and $k_2/A_2 = -3\Omega^2$. However, it also easily is shown that the bound is not attainable if the only velocity-dependent torques are from a viscous damper.

Note that the attainable bound of the two-body system considered in Ref. 1, consisting only of a linear spring and dashpot between the bodies, is $D = (3)^{1/2}\Omega/(5)^{1/4} = 0.67 \times (3)^{1/2}\Omega$. Hence, at best, a more sophisticated feedback control in this class of systems may increase from $D = 0.67 \times 3^{1/2}\Omega$ to $3^{1/2}\Omega$.

Degenerate third-order system

If $k_2 = a_2 = 0$, Eq. (2) is seen to degenerate; one of the roots becomes $p = 0$, and the others are the roots of a cubic. The root $p = 0$ implies that 1) the second body is in equilibrium at an arbitrary position, and 2) a constant torque applied to either body results in a constant velocity of the second body. When the second body is a fluid or solid flywheel as considered in Ref. 5, probably neither 1 nor 2 is a drawback, and the positions of the remaining, nonzero roots determine the decay rate. These are given by the roots of

$$A_1A_2p^3 + (c_1A_2 + c_2A_1)p^2 + (k_1A_2 + a_1A_2)p + c_2k_1 = 0 \qquad (7)$$

Applying inequality (3) to the constant term and the coefficient of p^2, one finds

$$D^2 \leq \frac{3c_2k_1}{c_1A_2 + c_2A_1} \qquad (8)$$

If it again is required that $c_1 \geq 0$, $c_2 \geq 0$, then (8) together with the restriction $k_1 \leq 3\Omega^2A_1$ results in

$$D \leq 3\Omega \qquad (9)$$

This is higher by a factor of $3^{1/2}$ than the bound (6) for the nondegenerate system but still of the order of the orbital rate.

It easily is verified that bound (9) is attained when $c_1 = 0$, $c_2/A_2 = 9\Omega$, $a_1/A_1 = 24\Omega^2$, and $k_1 = 3\Omega^2$.

III. System of Arbitrarily Rapid Decay Rate

The decay rate may be made arbitrarily large if the restriction to resistive velocity-dependent torques is removed. Consider the case of two crossed dumbbells, for which $k_1 = 3\Omega^2A_1$, $k_2 = -3\Omega^2A_2$. Suppose that Eq. (2) has a fourfold negative real root at $p = -D$. The coefficients of (2) are then in the ratio $1{:}4D{:}6D^2{:}4D^3{:}D^4$. From this it follows that

$$\begin{aligned} c_1/A_1 &= 2D[1 - (D^2/3\Omega^2)] \\ c_2/A_2 &= 2D[1 + (D^2/3\Omega^2)] \\ a_1/A_1 &= \tfrac{1}{2}[6D^2 - (D^4/3\Omega^2) - 3\Omega^2] \\ a_2/A_2 &= \tfrac{1}{2}[6D^2 + (D^4/3\Omega^2) + 3\Omega^2] \end{aligned} \qquad (10)$$

As $D \rightarrow \infty$, each of the parameter ratios of Eqs. (10) becomes infinite in magnitude. This means that for large D the system is extremely sensitive to changes in parameter values. For example, suppose for a given D that c_1/A_1 is in error by ϵ. Then the coefficient of p^3 in the characteristic Eq. (2) would be

$$\begin{aligned} c_1A_2 + c_2A_1 &= A_1A_2\left[2D\left(1 - \frac{D^2}{3\Omega^2}\right)\left(1 + \epsilon\right) + 2D\left(1 + \frac{D^2}{3\Omega^2}\right)\right] \\ &= 2DA_1A_2\left[2 + \epsilon\left(1 - \frac{D^2}{3\Omega^2}\right)\right] \end{aligned}$$

If for $\epsilon = 0.01$ the system were designed to attain $D = 30\Omega$, then one would have $c_1A_2 + c_2A_1 = 2DA_1A_2[2 - 0.01 \times 299] < 0$. Hence, a 1% error in this case would give a negative coefficient of p^3 and an unstable system.

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Exhaustion of Geomagnetically Trapped Radiation

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The amount of matter which must be injected into belts of geomagnetically trapped radiation in order to lower the radiation levels has been estimated. Two mechanisms are available for this purpose: the elastic scattering of particles into paths which will intersect the atmosphere before mirroring, and the removal of the particles' energy by ionization and excitation (i.e., the normal stopping power of matter).

The latter is easier to estimate. A particle of velocity v (cm/sec) has a path length of m(g/cm^2) in matter that is relatively independent of the stopping material. If a time t (sec) is allowed to stop it, the density of matter in the radiation belt must be m/vt (g/cm^3). For a belt of volume V (cm^3), one therefore must orbit a total mass $M = Vm/vt$ (g).

For the artificial radiation belt produced by recent nuclear tests, one may assume that the main component is electrons of approximately 1 Mev, so that m can be taken as 0.5 g/cm^2 and v as 3×10^{10} cm/sec. The belt volume is about 10^{26} cm^3, and one obtains $Mt = 1.7 \times 10^{15}$ g-sec or about 50 ton-yr.

For the protons of the inner Van Allen belt, m and V are slightly larger and v is slightly less, so that Mt is of the order of 300 ton-yr. Here, however, there is an upper limit on t which is given by the average natural residence time of the protons, which so far is unknown. The persistence of the artificial belt in a region of low natural radiation intensity suggests that residence times may be much longer and natural injection rates much lower than hitherto has been supposed. If this is so, a permanent reduction of radiation levels could be achieved in due course by the introduction of feasible amounts of matter. The reduction would proceed until particle concentrations are so low that the removal rate (natural and artificial) equals the arrival rate.

For the outer Van Allen belt, the values of m and v are comparable to those for the artificial belt, but V is considerably greater, so that Mt here is also of the order of 300 ton-yr. Again the maximum value of t is unknown.

In order to scatter trapped particles out of the belts, they must be deflected into a narrow cone along the magnetic lines of force. It is assumed that it would require at least 10 collisions of more than 10° each to effect this for the average belt particle. The number of atoms per cubic centimeter, N, which is required to produce this scattering in t sec is given by $N\sigma vt = 10$, where σ is the cross section for elastic scattering of the belt particles through an angle of 10° or more. The cross section can be calculated from the Coulomb scattering law.[1] For 1 Mev electrons scattered off a material with an average atomic number of 3 (e.g., a hydrocarbon), one obtains $\sigma = 2 \times 10^{-23}$ cm^2. This gives $Nt = 1.7 \times 10^{14}$ sec $\times$ particles/cm^3 or, taking a belt volume of 10^{26} cm^3 and an average atomic weight of 5, $Mt = 450$ ton-yr. This is therefore a less efficient exhaustion mechanism than energy removal.

The cross section for elastic scattering is inversely proportional to the square of the particle kinetic energy. The elastic scattering mechanism is therefore even less efficient for the highly energetic particles of the inner Van Allen belt, when compared to the energy removal mechanism which is most efficient for heavy particles. In the outer Van Allen belt, the ratio of efficiencies for the two mechanisms is comparable to that in the artificial belt.

It should be noted that, because of the strong energy dependence of σ, the relative efficiency of the scattering process increases as the particle energy goes down. The two mechanisms therefore reinforce each other. Some consequences of these calculations are as follows.

It does not seem desirable or quite possible at present to orbit matter with the exclusive or primary purpose of removing the artificial or natural radiation belts. Objects already in orbit are performing this function rather slowly and inefficiently, since their configuration is not optimal for this purpose. With the future establishment of space stations on orbits intersecting the belts, it may become very desirable to sweep the belts out in the manner indicated. The space stations themselves would not be good for this purpose because of the personnel radiation doses that would be absorbed from bremsstrahlung.

The choice of material for a sweepout program is not particularly critical, but the physical state is quite important. A gas would dissipate too rapidly, whereas solid particles in the micron to centimeter size range would increase unduly the micrometeorite hazard. Large chunks of materials are inefficient as particle exhaustors because most paths through them are longer than the distance required to stop a particle. This leaves two efficient forms of matter distribution: as a colloid or as thin sheets. Optical scattering effects of the colloidal material may be objectionable for astronomical observations. The authors consider that the best way to orbit material for belt exhaustion would be as thin sheets of solid material. Aluminum would be very suitable because it is notably stable to ionizing radiation. The interference with astronomical observations would be minimal; 10 tons of 0.05-g/cm^2 sheet would have a maximum area of 20,000 m^2, or 10^{-4} sterad at 500 km.

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